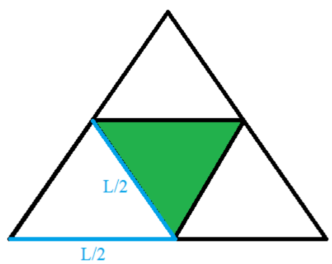
Problem 15

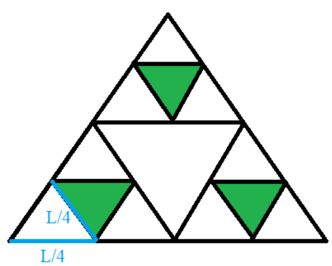
Connect the midpoints of the sides of an equilateral triangle to form 4 smaller equilateral triangles. Leave the middle small triangle blank, but for each of the other 3 small triangles, draw lines connecting the midpoints of the sides to create 4 tiny triangles. Again leave each middle tiny triangle blank and draw the lines to divide the others into 4 parts. Find the infinite series for the total area left blank if this process is continued indefinitely. (Suggestion: Let the area of the original triangle be 1; then the area of the first blank triangle is 1/4.) Sum the series to find the total area left blank. Is the answer what you expect? *Hint:* What is the "area" of a straight line? (Comment: You have constructed a *fractal* called the Sierpiński gasket. A fractal has the property that a magnified view of a small part of it looks very much like the original.)

Solution

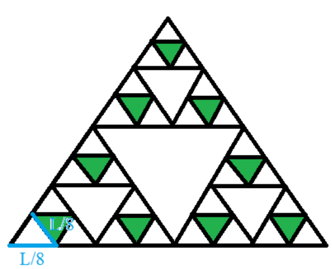
Consider an equilateral triangle that has a side length of L. Draw the lines connecting the midpoints.



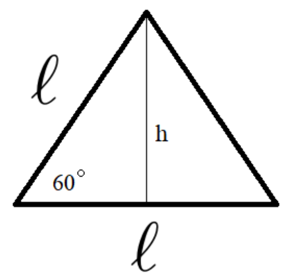
Draw the lines connecting the midpoints of these smaller triangles.



Draw the lines connecting the midpoints of these even smaller triangles.



The aim in this problem is to find the sum of the green areas in the limit as the number of triangles goes to infinity. Start by finding the area of an equilateral triangle.



$$A = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(\ell)(\ell \sin 60^\circ) = \frac{\ell^2}{4}\sqrt{3}$$

Every triangle in the Sierpiński gasket gives rise to three more, so the number grows exponentially: It goes from 1 to 3 to 9 .... The number of triangles with a certain side length is fixed. Let G be the sum of the areas of all the green triangles, let A_n be the area of a triangle with a certain side length, and let F_n be the number of triangles with a certain side length.

$$\begin{aligned}
G &= \sum_{n=1}^{\infty} A_n F_n \\
&= \sum_{n=1}^{\infty} \left(\frac{\ell_n^2}{4}\sqrt{3}\right)(3^{n-1}) \\
&= \sum_{n=1}^{\infty} \left[\frac{\sqrt{3}}{4}\left(\frac{L}{2^n}\right)^2\right](3^{n-1}) \\
&= \sum_{n=1}^{\infty} \left[\frac{\sqrt{3}}{4}\left(\frac{L^2}{4^n}\right)\right]\left(\frac{3^n}{3}\right) \\
&= \frac{L^2\sqrt{3}}{12}\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n \\
&= \frac{L^2\sqrt{3}}{12}\left[-1 + \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n\right] \\
&= \frac{L^2\sqrt{3}}{12}\left[-1 + \frac{1}{1-\left(\frac{3}{4}\right)}\right] \\
&= \frac{L^2\sqrt{3}}{12}(3) \\
&= \frac{L^2}{4}\sqrt{3}
\end{aligned}$$

The sum of all the green areas is the area of the original equilateral triangle.